Separation of Variables

Example

Solve the following differential equation: $\dfrac{dy}{dx} = y^2 \cos x$

Fact — In general, if we have a differential equation of the form $\dfrac{dy}{dx} = \dfrac{P(x)}{Q(y)}$ then we can solve it by equating $\displaystyle\int Q(y)dy = \int P(x)dx$

Example

Solve the differential equation $x^3 \dfrac{dy}{dx} + 3x^2 y = x^4$

Linear First Order Differential Equations

Fact — If we have a differential equation of the form

$$f(x)\frac{\mathrm{d}y}{\mathrm{d}x} + g(x)y + h(x) = 0,$$

then it is a **linear**, **first order**, **differential equation**

Definition. It is **linear** because

Definition. It is **first order** because

Fact — The **standard form** of a first order linear differential equation is

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$$

Example

Find the general solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x} = x$

Integrating Factors

Fact — To find the general solution of the differential equation $f(x)\dfrac{\mathrm{d}y}{\mathrm{d}x} + g(x)y = h(x)$ using an integrating factor:

Step 1: If $g(x) = f'(x)$, write the equation as $\frac{\mathrm{d}}{\mathrm{d}x}(y \cdot f(x)) = h(x)$ and go to **Step 6**.

Step 2: Divide the equation by $f(x)$ to obtain the standard form $\dfrac{\mathrm{d}y}{\mathrm{d}x} + y \cdot p(x) = q(x)$.

Step 3: Find the simplest integral of $p(x)$; denote it by $I(x)$.

Step 4: Write $u(x) = e^{I(x)}$, and simplify this if possible. This is the **integrating factor**.

Step 5: Multiply the equation (in its form after Step 2) by $u(x)$, and write the equation as $\dfrac{\mathrm{d}}{\mathrm{d}x}(y \cdot u(x)) = q(x) \cdot u(x)$.

Step 6: Integrate the equation with respect to x, including an arbitrary constant.

Step 7: Put the solution into the form $y = \ldots$ by dividing by the function which multiplies y (that is $u(x)$ or $f(x)$).

Example

Find the general solution of $\frac{\mathrm{d}y}{\mathrm{d}x}\cos x + y \sin x = \tan x$

Example

Find the curve through $(0, 1)$ whose equation satisfies the differential equation $y' + y = e^x$